Introduction to the mathematics of quantum economics
by David Orrell

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(1) Use mathematics as a shorthand language, rather than an engine of inquiry. (2) Keep to them till you have done. (3) Translate into English. (4) Then illustrate by examples that are important in real life. (5) Burn the mathematics.
Alfred Marshall, 1906[1]

Too large a proportion of recent "mathematical" economics are merely concoctions, as imprecise as the initial assumptions they rest on, which allow the author to lose sight of the complexities and interdependencies of the real world in a maze of pretentious and unhelpful symbols.
John Maynard Keynes, 1936[2]

Nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical.
Richard Feynman, 1981[3]

Introduction

This document gives a technical introduction to some of the mathematics used in quantum economics, and is intended as a supplement for the book *Quantum Economics: The New Science of Money*. As the quotes above point out, economics is not the same as a mathematical proof; however quantum mechanics *is* mathematical, so to fully exploit its ideas some mathematics is useful (even if it is burned afterwards). The aim here is to sketch out the way in which the economy can be represented mathematically using the quantum formalism, show the advantages over the classical approach, and clarify (at least for those with some knowledge of basic matrix algebra) what it means to say that the economy can be treated as a quantum system in its own right.

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[1] "(6) If you can't succeed in 4, burn 3. This last I do often." Letter to A.L. Bowley, 27 February 1906.
[2] From *The General Theory of Employment, Interest and Money*.
[3] From a 1981 talk "Simulating Physics with Computers" on the idea of a quantum computer.

The quantum approach to economics is inspired by the empirical fact that the monetary system shows quantum properties such as discreteness, indeterminacy, entanglement, and so on. To borrow Feynman's expression, a simulation had therefore better be quantum mechanical too, in the sense that it reflects these properties (even if it doesn't directly use a quantum formalism). The point is therefore not that the quantum approach will be the best technique to model every aspect of the economy, but rather that the economy has quantum properties which may need to be taken into account (explicitly or implicitly) depending on the context.

Models are ultimately justified by their success at explaining and predicting data. While the focus here is on theory, it should be noted that the areas of quantum cognition and quantum finance are heavily empirical, basing their results on experimental data for the former, and market data for the latter. The broader area of quantum economics – dealing as it does with emergent properties of a complex system – incorporates in addition a variety of complexity-based techniques, from agent-based models to systems dynamics, which have also been empirically tested (an exception is quantum agent-based models, which to my knowledge have yet to be developed for economics). For details, please see the book, and the references therein.

1. Some basics

Perhaps the most basic mathematical tool in quantum theory is the concept of the Hilbert space, which is named for the German mathematician David Hilbert (1862-1943). It was developed as an abstract mathematical object in the first decade of the twentieth century, and was later adopted by researchers in quantum physics. Social scientists are now following their lead by applying it to problems in areas such as decision-making and finance, as seen below.[4]

A Hilbert space H is a type of vector space whose elements, denoted $|u\rangle$, have coefficients that can be complex numbers. The dual state $\langle u|$ is the complex conjugate of the transpose of

[4] Some researchers in cognitive science prefer to treat the Hilbert space as just a tool, and see the word "quantum" as a distraction. Irving Fisher, in his 1892 book *Mathematical Investigations in the Theory of Value and Prices*, had a similar problem with the word "utility" which he described as "the heritage of Bentham and his theory of pleasures and pains. For us his *word* is the more acceptable, the less it is entangled with his *theory*" (p. 23). Personally I think it would be a little forced to ignore the theory's connections with physical reality.

$|u\rangle$. The inner product between two elements $|u\rangle$ and $|v\rangle$ is denoted $\langle u|v\rangle$, and is analogous to the dot product in a normal vector space, with the difference that the result can again be complex. The outer product is denoted $|u\rangle\langle v|$, and is like multiplying a column vector by a row vector, which yields a matrix. The magnitude of an element $|u\rangle$ is given by $\sqrt{\langle u|u\rangle}$, and two elements are orthogonal if $\langle u|v\rangle = \langle u|v\rangle = 0$. The Hilbert space can therefore be viewed as a generalisation of Euclidean space, with the difference that there can be an infinite number of dimensions (though conditions apply), the basis need not be simple column vectors, and coefficients can be complex.

An operator $\hat{A}$ is a map which sends one element $|u\rangle$ of *H* to another element $\hat{A}|u\rangle$ of *H*. For example, the projection operator is defined as $\hat{P}_u = |u\rangle\langle u|$, and $\hat{P}_u|v\rangle = |u\rangle\langle u|v\rangle$ gives the projection of *v* onto *u*. Operators $\hat{A}$ and $\hat{B}$ do not generally commute, so $\hat{A}\hat{B} \neq \hat{B}\hat{A}$. A state $|u\rangle$ is an eigenvector of $\hat{A}$ if $\hat{A}|u\rangle = \lambda|u\rangle$ where λ is the associated eigenvalue. A key feature of quantum theory is that observables such as a particle's position or momentum are represented by Hermitian operators, which have real eigenvalues.[5] Instead of being passive elements, as in classical theory, they are operators that ask a question of the system. During a measurement of an observable, the system state $|S\rangle$ collapses to one of the eigenvectors of the associated operator, with a probability given by the square of the projection of the state $|S\rangle$ on that eigenvector.

To see the difference between the classical and quantum approaches, suppose that a person has a choice between a certain number of possible options. In classical probability theory, each choice *u* would be treated as a subset of the set *U* consisting of all choices. A person's cognitive state is represented by a function *p* with the probability of choosing *X* given by *p*(*u*). As a simple example, *U* could consist of two choices *u* and *v*, with respective probabilities *p*(*u*) and *p*(*v*), that satisfy *p*(*u*) + *p*(*v*)= 1.

In quantum cognition, a choice in response to a particular question is treated instead as an element (e.g. vector) $|u\rangle$ of a Hilbert space *H*, and a person's cognitive state is represented by an element $|S\rangle$, both of length 1. (The state $|S\rangle$ is sometimes called a wave function, although here it is static rather than time-varying.) Here the associated operator $\hat{P}_u$ is the one that

[5] A Hermitian operator is one which equals its Hermitian conjugate, which for a matrix operator is defined as the complex conjugate of the transpose, so $A = A^{\dagger} \equiv (A^T)^*$.

projects vectors onto the vector $|u\rangle$. The probability of the answer to the question being $|u\rangle$ is then given by the magnitude of the projection squared, which is $|\langle u|S\rangle|^2$.

This shift, from sets of elements to geometric projections, allows for more complicated probabilistic effects such as non-commutativity and interference, which are characteristic of human cognition. For example, projecting onto $|u\rangle$, and then onto $|v\rangle$, may not give the same result as when the order is reversed, which compares with the "order effect" in surveys (for a worked example, see the book's Appendix). The Hilbert space therefore appears to be the natural framework for simulating cognitive phenomena, and researchers have amassed a considerable number of empirical findings to back up that claim.[6]

2. The quantum market

Just as a person's cognitive state can be simulated as a member of a Hilbert space, so we can do something similar for the economy as a whole. As a starting point, we will consider a simplified financial market. I will follow here the approach described by the late Rutgers theoretical physicist Martin Schaden in a 2002 paper on quantum finance, see that paper for details and applications.[7]

Suppose that the market consists of a collection of agents (investors) $j = 1, 2, ..., J$ who buy and sell assets of types $i = 1, 2, ..., I$. Each agent holds cash (or debt) x^j. The market can be represented as a Hilbert space H, with the basis

$$B := \{|x^j, \{n_i^j(s) \geq 0, i = 1,..,I\}, j = 1,..,J >\}.$$

Here $n_i^j(s)$ is the number of assets i with a price of s dollars that are held by investor j.

An individual basis state represents a market where the price of every security, and the cash position of each agent, is known precisely. The basis states are orthogonal in the sense that if the market is in the state $|m\rangle$ then it cannot be in a different state $|n\rangle$, so if $m \neq n$ then the

[6] For a survey, see: Bruza, Peter D. et al. (2015), 'Quantum cognition: a new theoretical approach to psychology', *Trends in Cognitive Sciences* 19(7), pp 383 – 393.
[7] Schaden, M. (2002), 'Quantum finance', *Physica A* 316(1), pp. 511-538. See also: Bagarello, F. (2009), 'A quantum statistical approach to simplified stock markets', *Physica A* 388(20):4397-4406.

inner product $\langle m|n\rangle = 0$. In general the market state (wave function) *M* is never known this accurately and is instead represented by the linear superposition of basis states $|n\rangle$ in *B*:

$$|M\rangle \ = \ \sum_n A_n|n\rangle.$$

where the A_n are complex numbers, and $w_n = |A_n|^2$ is the probability that the market is in the state $|n\rangle$.

The phases of the A_n are left unspecified at this stage, but are key to understanding effects such as interference. As in quantum physics, these effects are seen more easily when considering individual transactions. The propensities of each agent to buy or sell an asset can themselves be modelled as quantum phenomena, which as discussed above experience interference effects, and these can interact to affect the market as a whole.

If we define the ground state $|0\rangle$ to be a market where agents hold no assets including cash, then we can build up a real market by transferring cash and assets to agents. Shares are added or removed from an agent's account by the use of the creation operator $\hat{a}_i^{\dagger j}(s)$ and the annihilation operator $\hat{a}_i^j(s)$. These operators are like those used in many-body quantum mechanics to simulate the behaviour of a collection of bosons.

Money creation is handled using a translation operator of the form

$$\hat{c}^{\dagger j}(s) = \exp\left(-s\frac{\partial}{\partial x^j}\right),$$

which increases the amount of cash held by agent *j* by *s* currency units. Similarly the Hermitian conjugate operator $\hat{c}^j(s) = \hat{c}^{\dagger j}(-s)$ lowers the cash holding of agent *j* by the amount *s*.

The buying and selling of one unit of an asset by agent *j* at price *s* is represented by the creation and annihilation operators in combination with cash transfers which reflect the exchange of money.

$$\hat{b}_i^{\dagger j}(s) = \hat{a}_i^{\dagger j}(s)\hat{c}^j(s),$$
$$\hat{b}_i^j(s) = \hat{a}_i^{\dagger j}(s)\hat{c}^{\dagger j}(s).$$

An important difference between cash and a security is that while money is a conserved quantity during transactions, price changes with time. After a security has been bought, it evolves into a superposition of states, each of different prices, with amplitudes specifying the probability of selling at that price.

While it might not be obvious from these dry equations, and we haven't considered factors such as the creation of money objects through the issuance of debt, money still has a very special (but usually understated) role in the quantum model. Unlike other assets, it has a stable defined price. Without money, it is impossible to assign a price to other assets in the first place. The fact that these assets have indeterminate value is what gives money its dualistic properties, combining as it does stable numbers and unstable values. While it isn't possible for an asset to have a negative price, an agent can have a negative amount of money. Finally, money is often created in the first place through loans, which lead to entanglement as discussed below.

We can build up an arbitrary market state from the vacuum state by using these operators to successively transfer cash and securities to each agent. To study how the market wave function evolves with time, we write

$$|M\rangle_{t'} = \hat{U}(t', t)|M\rangle_t,$$

where $\hat{U}(t', t)$ is a unitary linear operator, that can be viewed as rotating the hyperspace of all possible states in the Hibert space. The dynamical behaviour of the system is driven by a Hamiltonian $\hat{H}(t)$, which in physics corresponds to total energy. Over short periods of time, if we assume continuity, $\hat{H}(t)$ satisfies the Schrodinger equation

$$i\frac{\partial}{\partial t}|M\rangle_t = \hat{H}(t)|M\rangle_t.$$

However it is more realistic to consider discontinuous changes of the sort that occur with individual transactions, and develop Hamiltonians for things like cash flow, the trading of securities, and so on. As shown by Schaden and other researchers, these in turn can be used to derive global properties of markets, such as the probability distribution of expected returns.

The variables of the system can be loosely interpreted in terms of physical analogies. The price *s* of an asset (or more correctly its logarithm) is like position. As in physics, there is an uncertainty relation involving asset price, and the momentum of the price change. The creation of money or assets adds energy (as measured by the Hamiltonian) to the total energy of the system. The same techniques used to study many-body quantum systems can then be applied to make predictions about market behaviour.

Like quantum cognition, quantum finance has become a sizeable area of research, with many papers showing empirical results. If markets are assumed to be large and nearly efficient, then the results generally approximate those produced by the classical approach. (Indeed, researchers have so far largely tended to respect classical assumptions such as efficiency, in an attempt to replicate known results.) However quantum effects become more important for markets that are thinly traded. The quantum approach can also be used to describe markets driven by investor sentiment, where there is a significant degree of synchronisation between market participants.

To summarise, a market can be represented as a Hilbert space, in which the price of an asset is known precisely only at the time of a transaction. Ownership and context are important, so an asset purchased by one person at one price is distinct from the same asset purchased by another person at a different price. As in quantum cognition, the act of measuring an asset's price – in this case by buying or selling – has an effect on the price. By constructing an appropriate Hamiltonian equation, we can study the dynamics of market evolution. As in physics, the complexity of the system means that macro-level behaviour is often characterised by emergent properties that cannot be reduced to some lower level. Again this differs from the classical approach which assumes assets have a certain inherent value independent of context; money does not play an important role, other than as a metric; and calculations can be based on micro-foundations of individual utility optimisation.

While quantum finance concentrates on the specialised case of financial markets, and is used for studying the properties of assets such as stocks or bonds, the same methodology can in principle be extended to describe markets in general, and form the basis of a mathematical description of the quantum economy. Again, money has a special role as an asset with a fixed price, and the price of everything else is indeterminate until measured through monetary transactions.

3. Entanglement

As discussed in the book, a key advantage of the quantum approach in economics – but one which to my knowledge has not previously been addressed by researchers in quantum finance – is that it provides a natural framework for thinking about financial entanglement through loans and derivatives.

To first motivate the discussion, consider the physical example of a pair of entangled electrons, denoted 1 and 2, each of which has spin ½ when measured along a particular axis, but in opposite directions. The spin part of their wave function can be written as a superposition of two states:

$$|S\rangle = \frac{1}{\sqrt{2}}\,|1\uparrow\rangle|2\downarrow\rangle - \frac{1}{\sqrt{2}}|1\downarrow\rangle|2\uparrow\rangle,$$

where the arrow indicates the direction of spin of each electron.

The wave function tells us nothing about the direction of spin for either electron, only that they are opposite, so the total spin is zero. Now, suppose that we measure the spin for electron 1. We would then expect an equal chance of getting a positive or negative result. If it is the former, then the system must have collapsed to an eigenstate with positive eigenvalue, so is of the form

$$|S\rangle = |1\uparrow\rangle|2\downarrow\rangle.$$

A measurement of particle 2 can now yield only a negative result. The reason is that the wave function describes the system, including both particles, so a measurement on one is equivalent to a measurement on the system as a whole.

The financial version of entanglement can be expressed using a similar formalism. Instead of two entangled electrons, consider two people entangled by a loan contract; and instead of spin direction, we will use loan status (i.e. “default” or “no default”). As in quantum cognition, the debtor is modelled as initially being in a superposition of two states, with a decision acting as a measurement event. The loan status can therefore be expressed by a wave function of the form:

$$|S\rangle = \alpha\,|1\uparrow\rangle|2\downarrow\rangle - \beta|1\downarrow\rangle|2\uparrow\rangle,$$

Here α^2 and β^2 add to 1, and give the probability of default $|1\uparrow\rangle|2\downarrow\rangle$ and no default $|1\downarrow\rangle|2\uparrow\rangle$ respectively, so reflect the debtor’s propensity to default at a particular moment. If the debtor decides to default on the loan, that acts as a measurement on the system as a whole. At any time after that, if the creditor decides to assess the state of the loan, the result can only indicate default. The two parties are thus entangled.

Of course, systems can be correlated without any need to invoke quantum effects.[8] However the key point is that we are treating the debtor’s state regarding the loan as being in a superposition of the two states “default” and “no default”. The state of the loan is therefore indeterminate (we don’t know whether the debtor will default) yet still correlated, which is the definition of entanglement.

Another possible objection is that, after one of a pair of entangled particles has been measured, the second doesn’t need to check with the first to find out what its state is; while with a loan the creditor does. However the wave function equation applies to the loan agreement, which is an abstract thing that encompasses both parties. So from the point of view of that wave function (which again is what we are modelling) the state does change instantaneously; it is only measurements that take time. The difference between the physics version, and the financial version, then reduces to a question of the nature and reality of such wave functions, which would depend on one’s interpretation of quantum theory, and is a

[8] For example, suppose I have two beads, one red and one blue, and I give one to a friend without looking. Then if I check and find that I have the red one, I know that my friend has the blue one.

topic of debate for both physicists and social scientists.[9] But from a mathematical modelling perspective the two are the same.

One feature of the system is that, unlike for electrons, there is now only one axis of measurement. This means that the behaviour of a loan agreement is much less subtle than the physical version (though some social scientists do argue for rich versions of social entanglement); and also that it is not possible to reproduce Bell-type experiments, where entanglement is tested by changing the orientation of the axis. However Bell's experiments do not define entanglement, but were devised as a way to tease out entanglement for systems that cannnot be queried more directly. For loans, the entanglement is encoded by the terms of the agreement. Again, the equation applies only to the loan agreement, so default may for example be followed by a complex negotiation, but the same is true in a physical system where other forces can also come into play.

Since most money is produced through private bank lending, and the financial system is dominated by complex derivatives contracts, financial entanglement is a tremendously important part of the economy, yet one which has been largely neglected in mainstream models. A number of techniques are currently being developed to simulate collective decision-making using a quantum approach, and these could be used to model phenomena such as mass defaults, or the impact of collective behaviour on the generation of credit in an economy.[10]

4. Summary

The quantum approach has successfully been used to model the economy at both the level of individuals (in quantum cognition) and the level of markets (quantum finance). In either case, the state of the system is represented using a Hilbert space. Measurement procedures such as decisions and transactions take precedence over internal states such as known preferences or inherent values. The quantum approach therefore differs fundamentally from the classical one, and can be extended to offer an alternative model of the economy in general.

[9] A widely discussed example is whether Xantippe, the wife of Socrates, became a widow the instant her husband was forced to commit suicide, or only when she found out later. See: Wendt, A. (2015), *Quantum Mind and Social Science: Unifying Physical and Social Ontology* (Cambridge: Cambridge University Press), p. 194.

[10] One researcher investigating quantum models of collective decision making is Michael Schnabel: https://harris.uchicago.edu/directory/michael-schnabel

While the aim of this document is only to give an idea of how quantum techniques can be applied to the economy, the literature in this area is quite large and different researchers take different approaches. As shown by empirical results in quantum cognition, the quantum approach appears to be a natural fit for modelling human decision-making. And while quantum finance has so far failed to prove its advantages to most of the quantitative finance community, some traders have adopted the quantum methodology to understand and predict for example the behaviour of illiquid assets.

From the larger perspective of quantum economics, a main advantage of the quantum approach is that it naturally incorporates the dualistic properties of money. In classical economics, price is essentially equated with value (with allowances for “market failures”). In quantum mechanics, prices are seen as emerging from monetary transactions. One consequence is to sever the direct link between price and value. Another is to concentrate the modeller’s attention on the entangling properties of money.

While many people with a background in physics will be familiar with the quantum approach, and can easily apply methods from e.g. statistical mechanics to derive results, those trained in a classical approach may at first find it awkward or intimidating. However one of the main lessons of quantum economics is that, just because the economy emerges from quantum effects, this does not imply that quantum models are always needed. For example, there are no quantum macroeconomic models to compare with a Dynamic Stochastic General Equilibrium (DSGE) model; but just as weather forecasters don’t base their models on quantum physics, so it is possible to simulate the flow of money in a way that respects its complex emergent properties without needing to go down to the quantum level. Another method is hybrid models such as quantum agent-based models, that are similar to usual agent-based models, but model agent decisions as a quantum process. The quantum approach can also be used to rule out certain modelling approaches (including DSGE models which rely on assumptions such as equilibrium).

For more background and further reading, see davidorrell.com/quantumresources.html

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