Introduction to the mathematics of quantum economics

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(1) Use mathematics as a shorthand language, rather than an engine of inquiry. (2) Keep to them till you have done. (3) Translate into English. (4) Then illustrate by examples that are important in real life. (5) Burn the mathematics.

Alfred Marshall, 1906[1]

Too large a proportion of recent "mathematical" economics are merely concoctions, as imprecise as the initial assumptions they rest on, which allow the author to lose sight of the complexities and interdependencies of the real world in a maze of pretentious and unhelpful symbols.

John Maynard Keynes, 1936[2]

Nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical.

Richard Feynman, 1981[3]

1. Introduction

This document gives a technical introduction to some of the mathematics used in quantum economics, and is intended as a supplement for the book *Quantum Economics: The New Science of Money*. As the quotes above point out, economics is not the same as a mathematical proof, and the key ideas of quantum economics, such as the quantum theory of money and value, do not rely on equations. However the quantum formalism *is* mathematical, so to fully exploit its ideas some mathematics is useful (even if it is burned afterwards). The aim here is to sketch out the way in which the economy can be represented mathematically using the quantum formalism, show the advantages over the classical approach, and clarify

[1] "(6) If you can't succeed in 4, burn 3. This last I do often." Letter to A.L. Bowley, 27 February 1906.
[2] From *The General Theory of Employment, Interest and Money.*
[3] From a 1981 talk "Simulating Physics with Computers" on the idea of a quantum computer.

(at least for those with some knowledge of basic matrix algebra) what it means to say that the economy can be treated as a quantum system in its own right.

The quantum approach to economics is inspired by the empirical fact that the monetary system shows quantum properties such as discreteness, indeterminacy, entanglement, and so on. To borrow Feynman's expression, a simulation had therefore better be quantum mechanical too, in the sense that it reflects these properties (even if it doesn't directly use a quantum formalism). The point is therefore not that the quantum approach will be the best technique to model every aspect of the economy, but rather that the economy has quantum properties which may need to be taken into account (explicitly or implicitly) depending on the context.

Models are ultimately justified by their success at explaining and predicting data. While the focus here is on presenting the basic tools of the theory, and showing how they relate to the nature of economic transactions, rather than on specific results, it should be noted that the areas of quantum cognition and quantum finance are heavily empirical, basing their results on experimental data for the former, and market data for the latter. The broader area of quantum economics – dealing as it does with emergent properties of a complex system – incorporates in addition a variety of complexity-based techniques, from agent-based models to systems dynamics, which have also been empirically tested (an exception is quantum agent-based models, which to my knowledge have yet to be developed for economics). For details, please see the book, and the references therein.

An outline is as follows. Section 2 introduces the idea of the Hilbert space, and shows how quantum probability differs from its classical version using the example of human cognition. Section 3 discusses the quantization procedure for a dynamic system. In Section 4 this is applied to the paridigmatic example of the quantum harmonic oscillator, and it is shown how tools such as creation and annihilation operators can be used to build up a system of interacting bosons. Section 5 uses the same ideas to develop a quantum model of a market, where shares and cash now take the place of bosons. Section 6 outlines a model where market strategies are treated as quantum variables. Section 7 discusses the concept of entanglement, and Section 8 summarises the main conclusions. The document is under development so further sections will be added, please check back for updates.

2. Some basics

Perhaps the most basic mathematical tool in quantum theory is the concept of the Hilbert space, which is named for the German mathematician David Hilbert (1862-1943). It was developed as an abstract mathematical object in the first decade of the twentieth century, and was later adopted by researchers in quantum physics. Social scientists are now following their lead by applying it to problems in areas such as decision-making and finance, as seen below.[4]

A Hilbert space H is a type of vector space whose elements, denoted $|u\rangle$, have coefficients that can be complex numbers. The dual state $\langle u|$ is the complex conjugate of the transpose of $|u\rangle$. The inner product between two elements $|u\rangle$ and $|v\rangle$ is denoted $\langle u|v\rangle$, and is analogous to the dot product in a normal vector space, with the difference that the result can again be complex. The outer product is denoted $|u\rangle\langle v|$, and is like multiplying a column vector by a row vector, which yields a matrix. The magnitude of an element $|u\rangle$ is given by $\sqrt{\langle u|u\rangle}$, and two elements are orthogonal if $\langle u|v\rangle = \langle u|v\rangle = 0$. The Hilbert space can therefore be viewed as a generalisation of Euclidean space, with the difference that there can be an infinite number of dimensions (though conditions apply), the basis need not be simple column vectors, and coefficients can be complex.

An operator $\hat{A}$ is a map which sends one element $|u\rangle$ of H to another element $\hat{A}|u\rangle$ of H. For example, the projection operator is defined as $\hat{P}_u = |u\rangle\langle u|$, and $\hat{P}_u|v\rangle = |u\rangle\langle u|v\rangle$ gives the projection of v onto u. Operators $\hat{A}$ and $\hat{B}$ do not generally commute, so $\hat{A}\hat{B} \neq \hat{B}\hat{A}$. A state $|u\rangle$ is an eigenvector of $\hat{A}$ if $\hat{A}|u\rangle = \lambda|u\rangle$ where λ is the associated eigenvalue. For example $\hat{P}_u|u\rangle = |u\rangle\langle u|u\rangle = \lambda|u\rangle$, so $|u\rangle$ is an eigenvector of $\hat{P}_u$ with eigenvalue $\lambda = \langle u|u\rangle$. The expectation value of a linear operator $\hat{A}$ in the state $|u\rangle$ is given by $\langle u|\hat{A}|u\rangle$, i.e. the scalar product of $\langle u|$ with $\hat{A}|u\rangle$.

[4] Some researchers in cognitive science prefer to treat the Hilbert space as just a tool, and see the word "quantum" as a distraction. Irving Fisher, in his 1892 book *Mathematical Investigations in the Theory of Value and Prices*, had a similar problem with the word "utility" which he described as "the heritage of Bentham and his theory of pleasures and pains. For us his *word* is the more acceptable, the less it is entangled with his *theory*" (p. 23). Personally I think it would be a little forced to ignore the theory's connections with physical reality.

A key feature of quantum theory is that observables such as a particle's position or momentum are represented by Hermitian operators, which have real eigenvalues.[5] Instead of being passive elements, as in classical theory, they are operators that ask a question of the system. During a measurement of an observable, the system state $|S\rangle$ collapses to one of the eigenvectors of the associated operator, with a probability given by the square of the projection of the state $|S\rangle$ on that eigenvector.

To see the difference between the classical and quantum approaches, in the context of human cognition, suppose that a person has a choice between a certain number of possible options. In classical probability theory, each choice u would be treated as a subset of the set U consisting of all choices. A person's cognitive state is represented by a function p with the probability of choosing X given by $p(u)$. As a simple example, U could consist of two choices u and v, with respective probabilities $p(u)$ and $p(v)$, that satisfy $p(u) + p(v)= 1$.

In quantum cognition, a choice in response to a particular question is treated instead as an element (e.g. vector) $|u\rangle$ of a Hilbert space H, and a person's cognitive state is represented by an element $|S\rangle$, both of length 1. (The state $|S\rangle$ is sometimes called a wave function, although here it is static rather than time-varying.) Here the associated operator $\hat{P}_u$ is the one that projects vectors onto the vector $|u\rangle$. The probability of the answer to the question being $|u\rangle$ is then given by the magnitude of the projection squared, which is $|\langle u|S\rangle|^2$.

This shift, from sets of elements to geometric projections, allows for more complicated probabilistic effects such as non-commutativity and interference, which are characteristic of human cognition. For example, projecting onto $|u\rangle$, and then onto $|v\rangle$, may not give the same result as when the order is reversed, which compares with the "order effect" in surveys. For a worked example, see the book's Appendix, or the web application available at https://david-systemsforecasting.shinyapps.io/ordereffect/ (see screenshot below). The Hilbert space therefore appears to be the natural framework for simulating cognitive phenomena, and researchers have amassed a considerable number of empirical findings to back up that claim.[6]

[5] A Hermitian operator is one which equals its Hermitian conjugate, which for a matrix operator is defined as the complex conjugate of the transpose, so $A = A^{\dagger} \equiv (A^T)^*$.
[6] For a survey, see: Bruza, Peter D. et al. (2015), 'Quantum cognition: a new theoretical approach to psychology', *Trends in Cognitive Sciences* 19(7), pp 383 – 393.

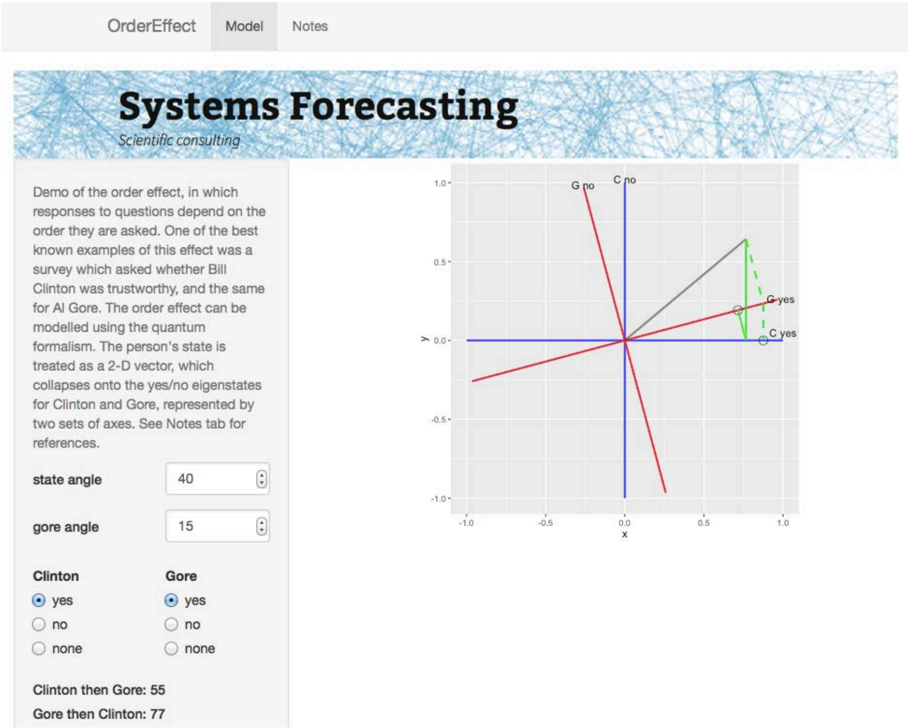

Figure shows a screenshot of the order effect demo, available as a web application.

3. Quantizing dynamics

As seen above, the key idea in the quantum approach is that point objects are replaced with quantum state or wave functions, and observables are replaced with the eigenvalues of operators. A typical question in physics is how a given set of equations can be “quantized” in this way.

One clue on how to go about this is the fact (first discovered by the mathematician Oliver Heaviside in the late nineteenth century) that differential operators act in some respects like ordinary numbers. Consider for example the equation

$$y + \frac{dy}{dx} = x^2.$$

Define D to be the differential operator $D = \frac{d}{dx}$, so $Dy = \frac{dy}{dx}$. Powers of D are interpreted as higher derivatives, so

$$D^2 = \frac{d^2}{dx^2}$$

and so on. Then the above equation can be written

$$(1 + D)y = x^2$$

so

$$y = \frac{x^2}{1 + D}.$$

Rewriting $\frac{1}{1+D}$ as the infinite expansion

$$\frac{1}{1 + D} = 1 - D + D^2 - D^3 \ldots$$

gives

$$y = (1 - D + D^2 - D^3 \ldots)x^2 = x^2 - 2x + 2$$

after applying the derivative operators to *x* and noting that all derivatives higher than the second are zero.

Because operators act on the object to the right of them, the two don't usually commute. Suppose we have a function $\psi(x)$ and evaluate

$$D(x\psi) = D(x)\psi + xD(\psi) = \psi + xD(\psi)$$

so

$$D(x\psi) - xD(\psi) = D(x)\psi + xD(\psi) = \psi$$

or in operator form

$$Dx - xD = 1$$

where 1 is the identity operator that does nothing. The commutator for two elements *f* and *g* is defined as $[f, g] = fg - gf$, so here we can write $[D, x] = 1$. Such commutator relationships play an important role in quantum mechanics. One has to be careful about the order of operations, and in quantizing a system it may not be clear at first which is the correct order to use.

Now, we want to represent quantum states using wave functions. Many experiments suggest waves that have a periodicity which scales with momentum, in a manner which depends on the reduced Planck's constant $\hbar$. Focussing on the spatial variation, a typical wave function might therefore be of the form

$$\psi(x) = e^{-\frac{ipx}{\hbar}}.$$

In classical mechanics x would refer to a spatial coordinate, and p to a momentum. If we identify $\hat{p}$ as the differential operator

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

and apply it to ψ we get

$$\hat{p}\psi = -i\hbar\frac{\partial\psi}{\partial x} = \hat{p}e^{-\frac{ipx}{\hbar}} = p\psi$$

so the observable *p* is an eigenvalue of the operator. We can therefore identify $\hat{p}$ as the momentum operator. The position operator $\hat{x}$ returns the value of *x*. In "momentum space" it can be defined as

$$\hat{x} = i\hbar\frac{\partial}{\partial p}$$

which has the eigenvalue *x*. A similar relationship (related to the requirements of relativity) holds for total energy $\hat{H}$ and time *t*:

$$\hat{H} = -i\hbar\frac{\partial}{\partial t}.$$

Using the definition of the momentum operator, and the product rule of calculus, we have

$$\hat{x}\hat{p}\psi - \hat{p}\hat{x}\psi = \hat{x}\left(-i\hbar\frac{\partial\psi}{\partial x}\right) + i\hbar\frac{\partial(\hat{x}\psi)}{\partial x}$$

$$= -\hat{x}\left(i\hbar\frac{\partial\psi}{\partial x}\right) + i\hbar\left(\hat{x}\frac{\partial\psi}{\partial x} + \frac{\partial\hat{x}}{\partial x}\psi\right) = i\hbar\frac{\partial\hat{x}}{\partial x}\psi.$$

But since $\frac{\partial\hat{x}}{\partial x} = 1$, it follows that $\hat{x}\hat{p}\psi - \hat{p}\hat{x}\psi = i\hbar\psi$, and the commutator therefore satisfies the relationship $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$. This is known as the canonical commutator relationship, which holds also for other pairs such as energy and time.

If the quantization procedure thus described seems a little ad hoc and awkward, one reason is that we are trying to adapt classical mathematical tools to handle wave/particle duality. Another is that the approach was based on intuition and the equations were adopted, not because they can be proved to be true, but because they fit the data (which gives some latitude for social scientists to adapt them for other uses). To get a better sense of how it works, we can apply the method to a simple physical example, which is the harmonic oscillator. We choose it because it plays a key role in quantum field theory, which underpins the methods used later to describe the quantum economy. Also it is one of the few quantum systems that can be solved in closed form equations.

4. The harmonic oscillator

A classical harmonic oscillator involves an object of mass *m* oscillating around a central point with a spring-like restoring force given by $F = -kx$, where *k* is a constant and *x* is the displacement. The energy E of a particle with position *x* and momentum *p* is then given by

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$$

where $\omega = \sqrt{k/m}$ is the frequency of oscillation. The first term represents the kinetic energy, and the second term the potential energy. The equation of motion can be determined by using Newton's law:

$$F = ma = m\frac{d^2y}{dx^2} = -kx$$

which has the oscillatory solution

$$x = A\cos(\omega t + \varphi)$$

where the phase φ depends on the starting point.

To quantize the system, we again need to replace the classical equations with quantum versions that act on wave functions but recover the required properties of observables.[7] In quantum mechanics, the total energy is given by an equation known as the Hamiltonian, expressed now in terms of operators. We therefore try:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

This can be written more simply in the form

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

where

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right),$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right),$$

$$\hat{N} = \hat{a}^\dagger\hat{a}.$$

For reasons that will become clear, $\hat{a}^\dagger$ is known as the creation operator, $\hat{a}$ is the annihilation operator, and $\hat{N}$ is the number operator. As seen by multiplying them out and using the commutator relationship between $\hat{x}$ and $\hat{p}$, the creation and the annihilation operators satisfy the canonical commutator relationship with this scaling, which is

[7] I am drawing on: Barton Zwiebach. 8.05 Quantum Physics II. Fall 2013. Massachusetts Institute of Technology: MIT OpenCourseWare, https://ocw.mit.edu. License: Creative Commons BY-NC-SA.

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1.$$

If ψ is a wave function with norm 1, then

$$\langle\psi|\hat{H}|\psi\rangle = h\omega\langle\psi|\hat{a}^\dagger\hat{a}|\psi\rangle = \hbar\omega\langle\hat{a}\psi|\hat{a}\psi\rangle + \frac{\hbar\omega}{2} \geq \frac{\hbar\omega}{2}$$

since any norm cannot be less than zero.

Now, suppose that $|E\rangle$ is a normalised energy state of the system. Since observables correspond to eigenvalues, it follows that $|E\rangle$ must be an eigenvector of the Hamiltonian operator, with associated eigenvalue E:

$$\hat{H}|E\rangle = E|E\rangle.$$

From this and the above inequality, we have

$$\langle E|\hat{H}|E\rangle = E\langle E|E\rangle = E \geq \frac{\hbar\omega}{2}.$$

The system therefore has a minimum energy level given by $\frac{\hbar\omega}{2}$.

Consider the two states defined as

$$|E_+\rangle = \hat{a}^\dagger|E\rangle,$$

$$|E_-\rangle = \hat{a}|E\rangle.$$

We first note that

$$[\hat{H}, \hat{a}^\dagger] = \hat{H}\hat{a}^\dagger - \hat{a}^\dagger\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a})\hat{a}^\dagger - \hat{a}^\dagger\hbar\omega(\hat{a}^\dagger\hat{a}) = \hbar\omega(\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{a})$$

since the contribution of the constant term in the Hamiltonian cancels out. Using the commutator relationship for creation and annihilation operators then gives

$$[\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger.$$

Similarly

$$[\hat{H}, \hat{a}] = \hbar\omega\hat{a}$$

and also

$$\hat{N}|E\rangle = \left(\frac{\hat{H}}{\hbar\omega} - \frac{1}{2}\right)|E\rangle = \hat{N}_E|E\rangle$$

where $\hat{N}_E = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}$ is the number operator eigenvalue associated with this energy state. Then

$$\hat{H}|E_+\rangle = \hat{H}\hat{a}^\dagger|E\rangle = \left([\hat{H}, \hat{a}^\dagger] + \hat{a}^\dagger H\right)|E\rangle = (\hbar\omega + E)\hat{a}^\dagger|E\rangle = (E + \hbar\omega)|E_+\rangle,$$

$$\hat{H}|E_-\rangle = \hat{H}\hat{a}|E\rangle = \left([\hat{H}, \hat{a}] + \hat{a}H\right)|E\rangle = (-\hbar\omega + E)\hat{a}|E\rangle = (E - \hbar\omega)|E_-\rangle$$

so the energy state with $E_+ = E + \hbar\omega$ and $N_{E_+} = N_E + 1$ has an increased energy level, while the energy state with $E_- = E - \hbar\omega$ and $N_{E_-} = N_E - 1$ has a decreased energy level.

The reason $\hat{a}^\dagger$ is called the creation operator, and $\hat{a}$ the annihilation operator, is that these operators raise or lower the energy by $\hbar\omega$ and the number operator by one. The creation operator can always be applied to raise the energy, but the annihilation operator can only be applied to energy levels above the base level, since energy cannot be negative.

The lowest base level can be found by assuming there is a non-trivial state $|E\rangle$ that is annihilated by $\hat{a}$, so $\hat{a}|E\rangle = 0$. Thus $\hat{a}^\dagger\hat{a}|E\rangle = N|E\rangle = 0$, which implies that this is the energy state with $E = \frac{\hbar\omega}{2}$ and $N_E = 0$. We can derive the equation for this state by acting with position *x*:

$$\langle x|\hat{a}|E\rangle = \sqrt{\frac{m\omega}{2h}}\left\langle x\middle|\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\middle|E\right\rangle = 0.$$

If we define the wave function $\psi_E(x) = \langle x|E\rangle$ and use the definition of $\hat{p}$ as a differential operator, then this gives

$$\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_E(x) = 0$$

or

$$\frac{d\psi_E}{dx} = -\frac{m\omega}{\hbar}x\psi_E$$

with solution

$$\psi_E(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

which is a Gaussian distribution centered at 0. The existence of this ground state reflects the uncertainty principle, in the sense that an oscillator with no energy can’t exist (because then we would know the energy is zero), and has no classical analogue. Higher energy states are more complicated, and can be determined by successively applying the creation operator.

Another way to express this is by using the number operator. Denote states as $|n\rangle$ with associated eigenvalue *n*, so $\hat{N}|n\rangle = n|n\rangle$. The ground state is $|0\rangle$ (which is not the same as the zero vector). The next state $|1\rangle$ is obtained by using the creation operator on $|0\rangle$,

$$|1\rangle = \hat{a}^\dagger|0\rangle$$

and

$$\hat{N}|1\rangle = \hat{a}^\dagger\hat{a}\hat{a}^\dagger|0\rangle = (\hat{a}^\dagger[\hat{a},\hat{a}^\dagger] + \hat{a}^\dagger\hat{a}^\dagger\hat{a})|0\rangle = \hat{a}^\dagger|0\rangle = 1$$

where we have used $[\hat{a},\hat{a}^\dagger] = 1$ and $\hat{a}|0\rangle = 0$. The equations for higher energy states can be derived recursively to give

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle.$$

The states $|n\rangle$ form an orthonormal basis, so any state can be described in terms of a linear combination of these states.

Another operator which will prove useful is the translation operator defined as

$$T_{x_0} = e^{-\frac{i}{\hbar}\hat{p}x_0}$$

which acts on a state $|\psi\rangle$ by moving it by an amount x_0. To see this, the expectation value of $\hat{x}$ in the state $|\psi\rangle$ is

$$\langle\hat{x}\rangle_\psi = \langle\psi|\hat{x}|\psi\rangle$$

and the expectation of $\hat{x}$ in the state $T_{x_0}|\psi\rangle$ is

$$\langle\hat{x}\rangle_{T_{x_0}|\psi} = \left\langle\psi\middle|T_{x_0}^\dagger\hat{x}T_{x_0}\middle|\psi\right\rangle = \left\langle\psi\middle|e^{-\frac{i}{\hbar}\hat{p}x_0}\hat{x}e^{\frac{i}{\hbar}\hat{p}x_0}\middle|\psi\right\rangle.$$

The expression involving brackets can be solved to give

$$\langle\hat{x}\rangle_{T_{x_0}|\psi} = \left\langle\psi\middle|\hat{x} + \frac{i}{\hbar}[\hat{p},\hat{x}]x_0\middle|\psi\right\rangle = \hat{x} + x_0$$

as expected.[8]

If the translation operator is applied to the ground state $|0\rangle$, then the new state is called a coherent state, and can be expressed in terms of creation and annihilation operators as follows:

$$|\hat{x}_0\rangle = T_{x_0}|0\rangle = \exp\left(-\frac{i}{h}\hat{p}x_0\right)|0\rangle = \exp\left(\frac{x_0}{\sqrt{2}d}(\hat{a}^\dagger - \hat{a})\right)|0\rangle,$$

or alternatively

$$|\alpha\rangle = \mathrm{D}(\alpha)|0\rangle$$

where $\alpha = \frac{x_0}{\sqrt{2}d}$, and

$$\mathrm{D}(\alpha) = \exp\alpha(\hat{a}^\dagger - \hat{a})\,|0\rangle$$

is known as the displacement operator.

[8] Using the Baker-Hausdorff identity $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}\left[\hat{A},[\hat{A},\hat{B}]\right] + \ldots$ where here all but the first two terms vanish.

Calculation shows that the total energy of the translated system is increased relative to that of the ground state by an amount $\frac{1}{2}m\omega^2{x_0}^2$ which makes sense since it corresponds to the classical potential energy of a particle on a spring stretched an amount x_0. However the system is not in a single energy state, but is of the form

$$|\hat{x}_0\rangle = \sum_{n=0}^{\infty} c_n|n\rangle.$$

The probability of obtaining an energy equal to E_n is $c_n^2 = \frac{\lambda^n}{n!}e^{-\lambda}$ which is a Poisson distribution with mean $\lambda = \frac{m\omega x_0^2}{2\hbar}$.

So far we have only viewed the system in a static sense. To study how the wave function $|\psi\rangle$ evolves with time, we write

$$|\psi\rangle_t = \hat{U}(t, t_0)|\psi\rangle_{t_0}$$

where $\hat{U}(t, t_0)$ is a unitary linear operator, that can be viewed as rotating the hyperspace of all possible states in the Hibert space. Taking derivative with respect to time gives

$$\frac{\partial}{\partial t}|\psi\rangle_t = \frac{\partial\hat{U}(t, t_0)}{\partial t}|\psi\rangle_{t_0}.$$

Using the fact (easily checked) that

$$\hat{U}(t_0, t) = \hat{U}^{-1}(t, t_0) = \hat{U}^{\dagger}(t, t_0)$$

then gives

$$\frac{\partial}{\partial t}|\psi\rangle_t = \frac{\partial\hat{U}(t, t_0)}{\partial t}\hat{U}^{\dagger}(t, t_0)|\psi\rangle_t.$$

Recalling that

$$\hat{H} = -i\hbar\frac{\partial}{\partial t}$$

gives the Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle_t = \hat{H}(t)|\psi\rangle_t$$

which can be solved in a similar manner as the classical version to show that operators satisfy the same oscillatory equations of motion.

To summarise, the quantum model predicts that the observed energy levels of a harmonic oscillator are equally spaced with an interval of $h\omega$ and a minimum value of $\frac{h\omega}{2}$. Prior to measurement, the system will be in a superposed state of the form $|\psi\rangle = \sum_n A_n |n\rangle$, where the A_n are complex numbers, and $w_n = |A_n|^2$ is the probability that the oscillator is in the state $|n\rangle$. The evolution of the state can be solved using the Schrödinger equation.

As a physical example of the harmonic oscillator, a diatomic molecule such as the hydrogen molecule H_2 can be viewed as two atoms connected by a spring. Experimental observations show that such molecules absorb and emit photons whose frequencies are multiples of the oscillator frequency, as expected. Many other physical systems, such as the vibration of molecules in a solid, can be similarly approximated as a quantum harmonic oscillator. Most importantly, it turns out that the equations describing electromagnetic fields in quantum physics are like those of a harmonic oscillator, with the particles corresponding to photons, and the ground state corresponding to the energy of empty space. It is this energy that fuels the appearance of "virtual photons" which communicate the electromagnetic force.

In finance, the quantum model has been used to simulate stock return distributions, based on the idea of a spring-like "reversion to the mean" force which attracts stock prices to a long-run equilibrium.[9] The Gaussian ground state of the oscillation is supplemented by higher energy levels which contribute the "fat tails" that characterise empirical distributions, and give a better fit to data than the Gaussian curve by itself (as is usually used). However like classical theory this approach assumes the existence of such an equilibrium in the first place.

What carries over in a more general sense is the idea of representing a quantum system as a collection of particles, that can be added, removed, or translated through the use of operators. Indeed, another interpretation of the quantum model – known as the Fock space representation – is to see the harmonic oscillator as representing, not a single particle, but a collection of *n* fictitious particles each with energy $\hbar\omega$. In this picture, the creation and annihilation operators are seen as adding and removing these particles. The ground or vacuum state $|0\rangle$ has no particles, $|1\rangle$ has a single particle, $|2\rangle$ has two, and so on. This method, known as second quantization, underpins the quantum field theory of relativistic particles, used for example to represent systems of bosons. But it can also be applied to things like assets, where here *n* refers to the number of units held.

[9] K. Ahn, M. Y. Choi, B. Dai, S. Sohn and B. Yang (2017), 'Modeling stock return distributions with a quantum harmonic oscillator', *EPL* 120(3) , 38003.

The other thing which carries over to economics is the different nature of classical and quantum systems. While the classical harmonic oscillator is just a weight bouncing around on a spring, where quantities such as position, momentum, and energy can be precisely calculated, the quantum version is better described in terms of potentiality. We can only calculate the probability that a measurement will yield a particular result; and the complexity of quantum behaviour means that even this can only be easily done for relatively simple systems. In economics, this puts a strong limit on how much can be gained from using reductionist methods.

For example, neoclassical economists thought of the price amounts of a commodity in terms of space, marginal utility as a force, and total utility as energy. If the marginal utility of a commodity k is assumed to be constant over the range of interest (which it might be for something like gold), then the equation for the total energy or utility in the system is the same as for a particle in a linear potential extended by an amount x. In this picture, a money object of value x would be represented as a point in commodity space located a distance x from the origin along the axis corresponding to money (or gold, since money wasn't usually treated as a thing in itself).

To quantize this system, the natural approach would be to replace price with a position operator, and replace utility with an appropriate Hamiltonian (though we have no need for utility as a separate concept). With the constant $\hbar$ set to reflect the discretisation of the currency, a quantum money object could then be treated as a quantum entity with a collapsed wave function whose energy is limited to discrete levels. However, the fact that the wave function is collapsed means that the energy has a single fixed value. Its dualistic nature is only realised during interactions. Other assets would show quite different behaviour, because their value is not fixed but is represented by a quantum wave function. The action of purchasing an asset with cash would therefore resemble the action of the translation operator on the quantum harmonic oscillator. Purchasing an asset with new cash boosts the energy of the system, but the probabilities of observing a particular value smear out over time, so the selling price is only determined at the exact time of transaction. This is made clearer in the next section, where the second quantization method is used to build up a representation of a quantum economic system, and money is handled using a translation operator.

5. The quantum market

In the examples above we have seen that a person's cognitive state, or the state of a quantum harmonic oscillator, can be simulated as a member of a Hilbert space. Furthermore, single particles that are in superposed states can be viewed, in a dual sense, as a collection of fictitious particles in single states. We can do something similar for the economy as a whole, and model it as a collection of interacting particles in a Hilbert space. As a starting point, we will consider a simplified financial market. I will follow here the approach described by the late Rutgers theoretical physicist Martin Schaden in a 2002 paper on quantum finance, see that paper for details and applications.[10]

Suppose that the market consists of a collection of agents (investors) j = 1, 2, ..., J who buy and sell assets of types i = 1, 2, ..., I. Each agent holds cash (or debt) x^j. The market can be represented as a Hilbert space H, with the basis

$$B := \left\{ |x^j, \left\{ n_i^j(s) \geq 0, i = 1, .., I \right\}, j = 1, .., J\rangle \right\}.$$

Here $n_i^j(s)$ is the number of assets i with a price of s dollars that are held by investor j.

An individual basis state represents a market where the price of every security, and the cash position of each agent, is known precisely. The basis states are orthogonal in the sense that if the market is in the state $|m\rangle$ then it cannot be in a different state $|n\rangle$, so if $m \neq n$ then the inner product $\langle m|n\rangle = 0$. In general the market state (wave function) M is never known this accurately and is instead represented by the linear superposition of basis states $|n\rangle$ in B:

$$|M\rangle = \sum_n A_n |n\rangle$$

where the A_n are complex numbers, and $w_n = |A_n|^2$ is the probability that the market is in the state $|n\rangle$.

The phases of the A_n are left unspecified at this stage, but are key to understanding effects such as interference. As in quantum physics, these effects are seen more easily when considering individual transactions. The propensities of each agent to buy or sell an asset can themselves be modelled as quantum phenomena, which as discussed above experience

[10] Schaden, M. (2002), 'Quantum finance', *Physica A* 316(1), pp. 511-538. See also: Bagarello, F. (2009), 'A quantum statistical approach to simplified stock markets', *Physica A* 388(20):4397-4406.

interference effects, and these can interact to affect the market as a whole. We return to this in the next section.

If we define the ground state $|0\rangle$ to be a market where agents hold no assets including cash, then we can build up a real market by transferring cash and assets to agents. The approach is the same as that used in many-body quantum mechanics to simulate the behaviour of a collection of bosons, so shares are added or removed from an agent's account by the use of the creation operator $\hat{a}_i^{\dagger j}(s)$ and the annihilation operator $\hat{a}_i^j(s)$. Money creation is handled using a translation operator of the form

$$\hat{c}^{\dagger j}(s) = \exp\left(-s\frac{\partial}{\partial x^j}\right)$$

which increases the amount of cash held by agent *j* by *s* currency units. Similarly the Hermitian conjugate operator $\hat{c}^j(s) = \hat{c}^{\dagger j}(-s)$ lowers the cash holding of agent *j* by the amount *s*.

While it might not be obvious from these dry equations, and we haven't considered factors such as the creation of money objects through the issuance of debt, money still has a very special (but usually understated) role in the quantum model. Unlike other assets, it has a stable defined price. Without money, it is impossible to assign a price to other assets in the first place. The fact that these assets have indeterminate value is what gives money its dualistic properties, combining as it does stable numbers and unstable values. While it isn't possible for an asset to have a negative price, an agent can have a negative amount of money. Finally, money is often created in the first place through loans, which lead to entanglement as discussed below.

The buying and selling of one unit of an asset by agent *j* at price *s* is represented by the creation and annihilation operators in combination with cash transfers which reflect the exchange of money:

$$\hat{b}_i^{\dagger j}(s) = \hat{a}_i^{\dagger j}(s)\hat{c}^j(s),$$

$$\hat{b}_i^j(s) = \hat{a}_i^{\dagger j}(s)\hat{c}^{\dagger j}(s).$$

We can build up an arbitrary market state from the vacuum state by using these operators to successively transfer cash and securities to each agent. To study how the market wave function evolves with time, we write

$$|M\rangle_t = \widehat{U}(t, t_0)|M\rangle_{t_0}$$

where $\widehat{U}(t, t_0)$ is a unitary linear operator. The dynamical behaviour of the system is driven by a Hamiltonian $\widehat{H}(t)$, which again satisfies the Schrödinger equation

$$i\frac{\partial}{\partial t}\big|M\rangle_t = \widehat{H}(t)\big|M\rangle_t.$$

It is then possible to develop Hamiltonians for things like cash flow, the trading of securities, and so on (although the mathematics is more complicated than for something like the harmonic oscillator). As shown by Schaden and other researchers, these in turn can be used to derive statistical properties of markets.

The variables of the system can be loosely interpreted in terms of physical analogies. The price s of an asset (or more correctly its logarithm) is like position. As in physics, there is an uncertainty relation involving asset price, and the momentum of the price change. The creation of money or assets adds energy (as measured by the Hamiltonian) to the total energy of the system. The same techniques used to study many-body quantum systems can then be applied to make predictions about market behaviour, either in closed form or by explicitly modelling each agent.

An important difference between cash and a security is that while money is a conserved quantity during transactions, price changes with time. After a security has been bought, it evolves into a superposition of states, each of different prices, with amplitudes specifying the probability of selling at that price. As an example, suppose that a particular investor initially has no shares in a particular company, and then acquires one share at time 0 for a price s_0.[11] By making a number of simplifying assumptions, and some rather involved computations, Schaden shows that the probability of selling the stock a time T later for price s follows a lognormal distribution which depends on the expected return and volatility of the stock.[12] This is a well-known empirical result, that can be derived from standard stochastic approaches, so serves primarily as a consistency check. However it only holds for

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[11] The initial state $|M_0\rangle$ can be written $|M_0\rangle = \hat{b}^\dagger(s_0)|\widetilde{M}_0\rangle$ where $\hat{b}(s_0)|\widetilde{M}_0\rangle = 0$. Here the indices for other stocks and investors have been repressed for clarity, and $\widetilde{M}_0$ is a state where the investor has no shares in the company (which is why the annihilation operator yields 0). At time T, the state evolves to $\big|M_T\rangle = \widehat{U}(t, t_0)\big|M_0\rangle$. The probability that the investor can sell the single share at a price s can be computed by looking at the product $\langle\widetilde{M}_T\big|\hat{b}(s)\big|M_T\rangle$, where $\widetilde{M}_T$ is again a state that is annihilated by $\hat{b}(s)$.

[12] The formula is $P_T(s|s_0) = \frac{1}{s\sigma\sqrt{2\pi T}} \exp\left[-\frac{\left(\ln\left(\frac{s}{s_0}\right)-\mu T\right)^2}{2\sigma^2 T}\right]$ where μ is expected return and σ is volatility.

intermediate time scales of a month or more, and again assumes that the market is near equilibrium. The quantum approach helps to explain how this model breaks down at shorter time scales, or for assets which are infrequently traded.

To summarise this section, a market can be represented as a Hilbert space, in which the price of an asset is known precisely only at the time of a transaction. Ownership and context are important, so an asset purchased by one person at one price is distinct from the same asset purchased by another person at a different price. As in quantum cognition, the act of measuring an asset's price – in this case by buying or selling – has an effect on the price. By constructing an appropriate Hamiltonian equation, we can study the dynamics of market evolution. As in physics, the complexity of the system means that macro-level behaviour is often characterised by emergent properties that cannot be reduced to some lower level. Again this differs from the classical approach which assumes assets have a certain inherent value independent of context; money does not play an important role, other than as a metric; and calculations can be based on micro-foundations of individual utility optimisation.

Like quantum cognition, quantum finance has become a sizeable area of research, with many papers showing empirical results. If markets are assumed to be large and nearly efficient, then the results do generally approximate those produced by the classical approach. (Indeed, researchers have so far largely tended to respect classical assumptions such as efficiency, in an attempt to replicate known results.) However quantum effects become more important for markets that are thinly traded, and the quantum approach can also be used to describe markets driven by investor sentiment, where there is a significant degree of entanglement between market participants.

While quantum finance concentrates on the specialised case of financial markets, and is used for studying the properties of assets such as stocks or bonds, the same methodology can in principle be extended to describe markets in general, and form the basis of a mathematical description of the quantum economy. Again, money has a special role as an asset with a fixed price, and the price of everything else is indeterminate until measured through monetary transactions.

6. Supply and demand

In the previous section assets were modelled as bosons. However agents don't just have assets, they also have strategies and intentions to buy or sell, which are collectively responsible for the market forces known as supply and demand. These intentions can be modelled in a similar way, through quantum game theory, with the difference that – as in quantum cognition – the bosons now stand for mental states rather than valuable assets (though as always with the economy, these things are related). Following Gonçalves & Gonçalves (2007)[13], we will consider a simple market involving a single stock. At each time step of length τ, new information is supplied which has equal probability of being positive or negative about the stock. Agents then decide on a strategy to buy or sell the asset, based on the current price and the new information. The price is then collapsed to a new level by a market maker. (Again, I will just sketch the model, see the references for details.)

The state where there are n_0 buyers and n_1 sellers is denoted $|n_0, n_1\rangle$, and the prices are determined by the market maker using a formula that accounts for the balance between these two numbers $S(k) = n_1 - n_0$. The logarithmic returns from time step $k-1$ to k are given by

$$R(k) = \ln P(k) - \ln P(k-1) \approx \frac{S(k)}{\lambda}$$

where λ is a normalizing liquidity parameter.[14]

Define the number operators $\hat{N}_0 = \hat{a}_0^\dagger \hat{a}_0$ and $\hat{N}_1 = \hat{a}_1^\dagger \hat{a}_1$, the market maker observable

$$\hat{S} = \hat{N}_1 - \hat{N}_0$$

$$\hat{S}|n_0, n_1\rangle = (n_1 - n_0)|n_0, n_1\rangle,$$

the returns observable

$$\hat{R} = \frac{1}{\lambda}\left(\hat{N}_1 - \hat{N}_0\right)$$

$$\hat{R}|n_0, n_1\rangle = \frac{1}{\lambda}(n_1 - n_0)|n_0, n_1\rangle,$$

and the activity observable $\hat{N} = \hat{a}_0^\dagger \hat{a}_0 + \hat{a}_1^\dagger \hat{a}_1$.

At each time step t_k the game is started in ground state $|0,0\rangle$ meaning that opinion about the stock is neutral before news appears. The state of the game at the next step, a time τ later, is

[13] Gonçalves, C.P. & Gonçalves, C. (2007), 'An Evolutionary Quantum Game Model of Financial Market Dynamics − Theory and Evidence'. https://papers.ssrn.com/sol3/papers.cfm?abstract_id=982086
[14] Farmer, D.J. (1998), 'Market Force, Ecology, and Evolution', *Research in Economics* 98-12-117e, Santa Fe Institute.

determined from a unitary operator $U_k(t_{k+1}, t_k)$ acting on the first state. We write this as the product of displacement operators

$$U_k(t_{k+1}, t_k)|0,0\rangle = D\big(\xi_0(k)\big)D\big(\xi_1(k)\big)|0,0\rangle = |\xi_0(k), \xi_1(k)\rangle$$

where

$$D\left(\xi_j(k)\right) = \exp\left(\xi_j(k)\big(\hat{a}_j^\dagger - \hat{a}_j\big)\right)$$

and

$$\xi_j(k) = -i\tau\mu_j(k).$$

We are therefore treating the creation of buyers or sellers as being precipitated by the displacement of the ground state of a quantum system by an amount specified by $\xi_j(k)$. As seen for the example of the quantum harmonic oscillator, when this is expressed in the energy level basis, it leads to a (here bivariate) distribution among different energy levels of the form

$$|\xi_0, \xi_1\rangle = \exp\left(-\frac{|\xi_0|^2 + |\xi_1|^2}{2}\right)\sum_{n_0,n_1}\frac{\xi_0^{n_0}\xi_1^{n_1}}{\sqrt{n_0!\,n_1!}}|n_0, n_1\rangle$$

where the step *k* has been repressed for clarity.[15] The displacement therefore has the effect of shifting the system from its ground state, to a spectrum of potential states of higher energy (i.e. potential numbers of buyers and sellers).

We can then calculate the probability of each state by looking at the transition amplitudes from one time step to the next, which are given by

$$\psi(n_0, n_1) = \exp\left(-\frac{|\xi_0|^2 + |\xi_1|^2}{2}\right)\frac{\xi_0^{n_0}\xi_1^{n_1}}{\sqrt{n_0!\,n_1!}}$$

Taking the square to calculate probabilities leads to the bivariate Poisson distribution for the occupation numbers

$$|P_{\xi_0,\xi_1}(n_0, n_1)\rangle = |\psi(n_0, n_1)|^2 = \frac{\exp(-|\xi_0|^2 - |\xi_1|^2)\,|\xi_0|^{2n_0}|\xi_1|^{2n_1}}{n_0!\,n_1!}.$$

The expected value of the returns observable at the end of the round is then given by

$$\begin{aligned}\langle\hat{R}\rangle = \langle\xi_0, \xi_1\left|\frac{1}{\lambda}\big(\hat{N}_1 - \hat{N}_0\big)\right|\xi_0, \xi_1\rangle &= \frac{1}{\lambda}\langle\xi_0, \xi_1|\hat{N}_1|\xi_0, \xi_1\rangle - \frac{1}{\lambda}\langle\xi_0, \xi_1|\hat{N}_0|\xi_0, \xi_1\rangle \\ &= \frac{1}{\lambda}(|\xi_1|^2 - |\xi_0|^2)\end{aligned}$$

and similarly for the market activity observable

[15] See http://homepage.univie.ac.at/reinhold.bertlmann/pdfs/T2_Skript_Ch_5.pdf

$$\langle \hat{N} \rangle = \langle \xi_0, \xi_1 \left| \frac{1}{\lambda}\left(\hat{N}_1 + \hat{N}_0\right) \right| \xi_0, \xi_1 \rangle = \frac{1}{\lambda}(|\xi_1|^2 + |\xi_0|^2).$$

Market behaviour therefore depends on these two variables ξ_0, ξ_1 which encode the agent behaviour, and should reflect trader psychology. This section of the model is therefore similar to other such behavioural models.[16] We first write

$$\left|\xi_j(k)\right|^2 = \tau^2 \mu_j^2(k)$$

to remove the time dependence and use the rule

$$\mu_0(k) = \mu_0(k-1) - V(k)$$
$$\mu_1(k) = \mu_1(k-1) + V(k)$$
$$V(k) = \omega_M(k)S(k-1) + \omega_N(k)\sigma(k).$$

Here $S(k)$ is interpreted as an indicator of market momentum, and $\sigma(k)$ is market news which randomly takes on the values ± 1. Note that setting $\mu_0(0) = \mu_1(0) = 0$ means that for all subsequent time steps $\mu_0(k) = -\mu_1(k)$. The average numbers of buyers and sellers will then be the same:

$$\langle \hat{N}_0 \rangle = \tau^2 \mu_0^2(k)$$
$$\langle \hat{N}_1 \rangle = \tau^2 \mu_1^2(k)$$
$$\langle \hat{R} \rangle = 0$$

and the expected return is zero, which is consistent with the idea, which goes back to Bachelier, that expected returns are not affected by news. This feature can be relaxed, and a drift term can also be added in order to give a positive expected return.

In general $V(k) < 0$ implies that sentiment about value is negative, and $V(k) > 0$ that it is positive. In the model we assume $\omega_N(k)$ is fixed, so response to news is the same, but market contagion is determined by the formula

$$\omega_M(k) = \bar{\omega} + \gamma \omega_M(k-1) + \beta r(k-1)\frac{\sigma(k)}{M}$$

where $\bar{\omega}$ is a base level, γ is a trend-following parameter (the signal is stronger if it was strong in the previous step), and $\beta > 0$ amplifies the news signal depending on whether or not it confirms the returns $r(k-1)$ over the previous period. One can play around with these parameters to model different types of investor behaviour. For simulations, typical model parameters are $\beta = 1$, $\tau = 0.1$, $\lambda = 2000$, $\bar{\omega} = 0.1$, $0 < \gamma \leq 0.1$, and $\sigma(k) = \pm M$ with

[16] For a discussion, see Sornette, D. & Zhou, W-X (2006), Importance of positive feedbacks and over-confidence in a self-fulfilling Ising model of financial markets, *Physica A* 370, 704-726.

equal probability to give zero drift (e.g. no growth). The model then simulates many features that are characteristic of real market data, including turbulence and multifractal behaviour (i.e. fluctuations with more than a single fractal dimension). Indeed the main conclusion is that the behaviour is completely chaotic and unpredictable. A demo of the model is available at: https://david-systemsforecasting.shinyapps.io/quantummarket/. If you hit the run button often enough, you get something that looks rather like a Bitcoin price chart (see screenshot below) which tells you something about the wildness of such markets.

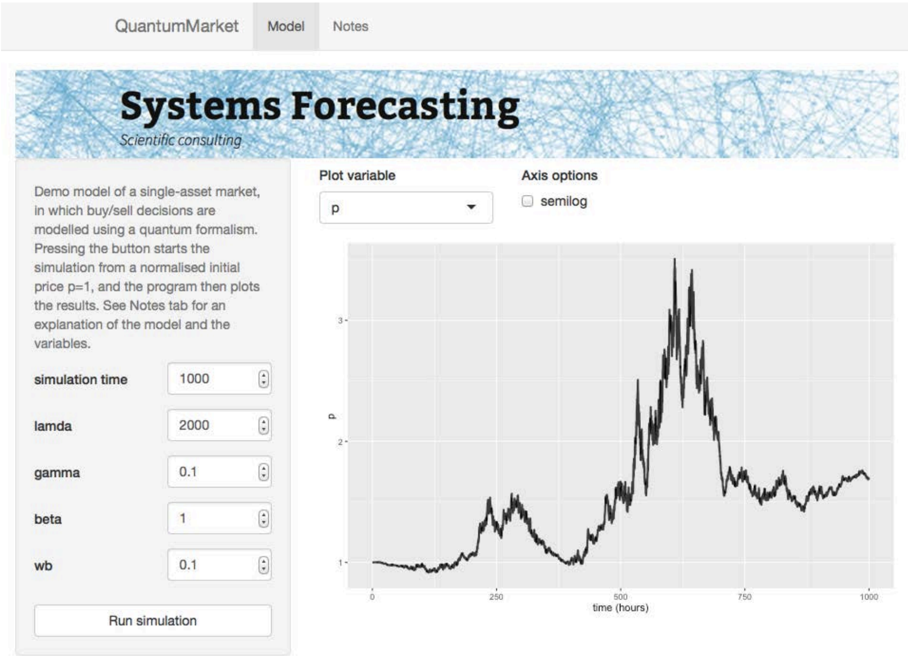

Figure shows a screenshot of the Quantum Market demo, available as a web application.

Because prices are collapsed at the end of each trading round, the main contribution of the quantum element in this model is to add a stochastic term to the equations determining the numbers of buyers and sellers. In many respects the model therefore resembles the stochastic models used in systems biology.[17] The quantum model would take on an extra level of complexity however if entanglement between classes of agents is considered, so that for example particular groups tend to align and react to news in the same way.

Perhaps the most interesting feature of the model from the perspective of quantum economics is that, while it simulates the collective propensity of agents to buy or sell, there is no

[17] Ramsey, S., Orrell, D., & Bolouri, H. (2005), 'Dizzy: stochastic simulation of large-scale genetic regulatory networks', *Journal of bioinformatics and computational biology* 3 (02), 415-436.

recourse to fixed curves of supply and demand. Instead these propensities tend to ramp up together during periods of high market intensity. Random news combine with quantum stochastic effects to create perturbations that are then amplified or damped through investor psychology and dynamics.

In some respects the model seems less neatly defined than Schaden's version from the previous section, where the equations represented the creation and annihilation of money and assets rather than the more nebulous notions of supply and demand. However, while the role of consciousness may be debated in physics, it is obviously integral to economics, since only a conscious entity can experience a sense of value. The fact that we use the same quantum formalism to simulate a monetary transfer, and a psychological transfer (e.g. in the interest in a stock), makes this connection concrete.

7. Entanglement

As discussed in the book, a key advantage of the quantum approach in economics – but one which to my knowledge has not previously been addressed by researchers in quantum finance – is that it provides a natural framework for thinking about financial entanglement through loans and derivatives.

To first motivate the discussion, consider the physical example of a pair of entangled electrons, denoted 1 and 2, each of which has spin ½ when measured along a particular axis, but in opposite directions. The spin part of their wave function can be written as a superposition of two states:

$$|S\rangle = \frac{1}{\sqrt{2}}\,|1\uparrow\rangle|2\downarrow\rangle - \frac{1}{\sqrt{2}}|1\downarrow\rangle|2\uparrow\rangle$$

where the arrow indicates the direction of spin of each electron.

The wave function tells us nothing about the direction of spin for either electron, only that they are opposite, so the total spin is zero. Now, suppose that we measure the spin for electron 1. We would expect an equal chance of getting a positive or negative result. If it is the former, then the system must have collapsed to an eigenstate with positive eigenvalue, so is of the form

$$|S\rangle = |1\uparrow\rangle|2\downarrow\rangle$$

A measurement of particle 2 can now yield only a negative result. The reason is that the wave function describes the system, including both particles, so a measurement on one is equivalent to a measurement on the system as a whole.

The financial version of entanglement can be expressed using a similar formalism. Instead of two entangled electrons, consider two people entangled by a loan contract; and instead of spin direction, we will use loan status (i.e. “default” or “no default”). As in quantum cognition, the debtor is modelled as initially being in a superposition of two states, with a decision acting as a measurement event. The loan status can therefore be expressed by a wave function of the form:

$$|S\rangle = \alpha\, |1\uparrow\rangle|2\downarrow\rangle - \beta|1\downarrow\rangle|2\uparrow\rangle$$

Here α^2 and β^2 add to 1, and give the probability of default $|1\uparrow\rangle|2\downarrow\rangle$ and no default $|1\downarrow\rangle|2\uparrow\rangle$ respectively, so reflect the debtor’s propensity to default at a particular moment. If the debtor decides to default on the loan, that acts as a measurement on the system as a whole. At any time after that, if the creditor decides to assess the state of the loan, the result can only indicate default. The two parties are thus entangled.

Of course, systems can be correlated without any need to invoke quantum effects.[18] However the key point is that we are treating the debtor’s state regarding the loan as being in a superposition of the two states “default” and “no default”. The state of the loan is therefore indeterminate (we don’t know whether the debtor will default) yet still correlated, which is the essence of entanglement.

Another possible objection is that, after one of a pair of entangled particles has been measured, the second doesn’t need to check with the first to find out what its state is; while with a loan the creditor does. However the wave function equation applies to the loan agreement, which is an abstract thing that encompasses both parties. So from the point of view of that wave function (which again is what we are modelling) the state does change instantaneously; it is only measurements that take time. The difference between the physics version, and the financial version, then reduces to a question of the nature and reality of such wave functions, which would depend on one’s interpretation of quantum theory, and is a

[18] For example, suppose I have two beads, one red and one blue, and I give one to a friend without looking. Then if I check and find that I have the red one, I know that my friend has the blue one.

topic of debate for both physicists and social scientists.[19] But from a mathematical modelling perspective the two are the same.

One feature of the system is that, unlike for electrons, there is now only one axis of measurement. This means that the behaviour of a loan agreement is much less subtle than the physical version (though some social scientists do argue for rich versions of mental entanglement based on physical principles); and also that it is not possible to reproduce Bell-type experiments, where entanglement is tested by changing the orientation of the axis. However Bell's experiments do not define entanglement, but were devised as a way to tease out entanglement for systems that cannnot be queried more directly. For loans, the entanglement is encoded by the terms of the agreement. Again, the equation applies only to the loan agreement, so default may for example be followed by a complex negotiation, but the same is true in a physical system where other forces can also come into play.

Since most money is produced through private bank lending, and the financial system is dominated by complex derivatives contracts, financial entanglement is a tremendously important part of the economy, yet one which has been largely neglected in mainstream models, precisely because they are based on a classical atomistic paradigm. A number of techniques are currently being developed to simulate collective decision-making using a quantum approach, and these could be used to model phenomena such as mass defaults, or the impact of collective behaviour on the generation of credit in an economy.[20]

8. Summary

The quantum approach has successfully been used to model the economy at both the level of individuals (in quantum cognition) and the level of markets (quantum finance). In either case, the state of the system is represented using a Hilbert space. Measurement procedures such as decisions and transactions take precedence over internal states such as known preferences or inherent values. The quantum approach therefore differs fundamentally from the classical one, and can be extended to offer an alternative model of the economy in general.

[19] A widely discussed example is whether Xantippe, the wife of Socrates, became a widow the instant her husband was forced to commit suicide, or only when she found out later. See: Wendt, A. (2015), *Quantum Mind and Social Science: Unifying Physical and Social Ontology* (Cambridge: Cambridge University Press), p. 194.

[20] One researcher investigating quantum models of collective decision making is Michael Schnabel: https://harris.uchicago.edu/directory/michael-schnabel

While the aim of this document is only to give an idea of how quantum techniques can be applied to the economy, the literature in this area is quite large and different researchers take different approaches. As shown by empirical results in quantum cognition, the quantum approach appears to be a natural fit for modelling human decision-making. And while quantum finance has not been widely adopted by the quantitative finance community, some traders have adopted the quantum methodology to understand and predict for example the behaviour of illiquid assets.

From the larger perspective of quantum economics, a main advantage of the quantum approach is that it naturally incorporates the dualistic properties of money. In classical economics, price is essentially equated with value (with allowances for "market failures"). In quantum mechanics, prices are seen as emerging from monetary transactions. One consequence is to sever the direct link between price and value. Another is to concentrate the modeller's attention on the entangling properties of money.

A natural extension of the market models considered above, and an interesting longer-term research project, would be a quantum agent-based model of something like a housing market. Following the approach used to model the propensity to buy or sell individual stocks (Section 6), each house could be considered as a separate single-asset market. Here the propensity to sell would model, not the number of sellers (since there is only one), but rather the pressure on the seller to complete a sale, which would affect the final price. Buyers and sellers would be entangled to a degree with each other, and to the news flow which could be modelled as a quantum variable. Such a model could simulate the kind of market contagion seen in housing markets, such as "fear of missing out" when prices are rising. It could also include the process of money creation through private lending, which grows the money supply and leads to asset price inflation.

While many people with a background in physics will be familiar with the quantum approach, and can easily apply methods from e.g. statistical mechanics to derive results, those trained in a classical approach may at first find it awkward or overly elaborate. However one of the main lessons of quantum economics is that, just because the economy *emerges* from quantum effects, this does not imply that quantum models are always obligatory. The complex behaviour of water, which ultimately arises from quantum properties, may drive the weather system, but weather forecasters don't base their models on quantum physics. Similarly it is possible to simulate the flow of money in a way that respects its complex emergent properties without needing to go down to the quantum level. The quantum approach can also be used to rule out certain modelling approaches, including Dynamic Stochastic General Equilibrium models (the so-called workhorses of macroeconomics), which rely on classical assumptions such as equilibrium.

For more background and further reading, see davidorrell.com/quantumresources.html
Contact the author at: futureofeverything.wordpress.com/about/